e.g. matrix Lie groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $O(n)$, $U(n)$, $Sp(2n)$, ...

(identify them with subsets in $\mathbb{R}^m$ and matrix product $\Leftrightarrow$ smooth maps in $\mathbb{R}^m$).

<u>Fact</u> (Ado-Iwasawa) Every connected Lie group can be "almost embedded" into $GL(n,\mathbb{R})$ for some n.

e.g. smooth maps between Lie groups

- Fix $g \in G$. $g\cdot : G \to G$. (multiplication by g <u>on the left</u>)
- Fix $g \in G$, $c(g) : G \longrightarrow G$ $\quad x \longrightarrow g\cdot x\cdot g$ $\quad c(g) \cdot x = gxg^{-1}$
- $F: G \to G'$ smooth map (between mfds) + group homomorphism

Such F is called a <u>Lie group homomorphism</u>

<u>Rmk</u>. Lie group homomorphism opens a door transferring from geo to alg.
(local-determining theorem, later).

⑥ reduction (from a Lie group action).

- Group action $G \curvearrowright M$ means an assignment $\sigma : G \to \mathrm{Diff}(M)$
  s.t. $\forall g, h \in G$, $\quad \sigma(h \circ g) = \sigma(h) \circ \sigma(g)$

  In particular, if G is a Lie group, we also require $G \times M \to M$ by

$$(g, m) \longmapsto \sigma(g)(m) \quad (\text{or simply } g \cdot m)$$

  is a smooth map.

e.g. $\mathbb{R}^* \curvearrowright \mathbb{R}^{n+1} \setminus \{0\}$ $\qquad \lambda \in \mathbb{R}^* \xrightarrow{\sigma} \sigma(\lambda) \in \mathrm{Diff}(\mathbb{R}^{n+1} \setminus \{0\})$

$$\sigma(\lambda) = \text{multiply by } \lambda.$$

$$(\lambda, (x_1, \cdots, x_{n+1})) \longmapsto (\lambda x_1, \cdots, \lambda x_{n+1}) \text{ is a smooth map.}$$

e.g. $S^1 \curvearrowright S^{2n+1}$ $\qquad \theta$ or $e^{2\pi i \theta} \in S^1 \xrightarrow{\sigma} \sigma(\theta) \in \mathrm{Diff}(S^{2n+1})$

$$S^{2n+1} = \{x_1^2 + \cdots + x_{2n+2}^2 = 1\}$$

$z_i = x_i + x_{i+n}\sqrt{-1}$ $\to$ $\{|z_1|^2 + \cdots + |z_{n+1}|^2 = 1\}$

$$\sigma(\theta) = \sigma(\theta)(z_1, \cdots, z_{n+1}) = (e^{2\pi i \theta} z_1, \cdots, e^{2\pi i \theta} z_{n+1})$$

- Ideally, group action $G \curvearrowright M$ enables to split M into the following structure

$$M = \bigcup_{x \in M} \underline{\hspace{6em}} \swarrow G\cdot\{x\} = \text{orbit space of } x \approx G$$

In other words, M splits into many copies of G.

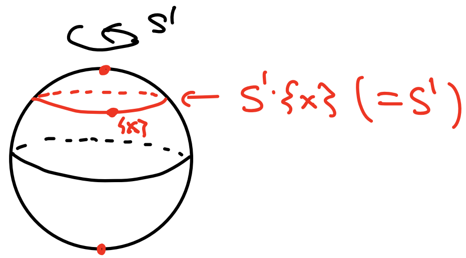
Good e.g.: $\mathbb{T}^n \curvearrowright \mathbb{C}^n \setminus \{0\}$ $\quad (\theta_1, \cdots, \theta_n)\cdot(z_1, \cdots, z_n) =: (e^{2\pi i \theta_1} z_1, \cdots, e^{2\pi i \theta_n} z_n)$

(angular-momentum coordinate)

$$\Rightarrow \mathbb{C}^n \approx \mathbb{T}^n \times \mathbb{R}^n_{>0}$$

$(|z_1|, \cdots, |z_n|)$ cf. polar coordinate.

Bad e.g. : $S^1 \curvearrowright S^2$



Some degeneration occurs at both north pole and south pole.

- $G \curvearrowright M$ is called <u>free</u> if $\forall\, x \in M$, the stabilizer of x satisfies

$$G_x = \{ g \in G \mid g \cdot x = x \} = \{e\}$$

Good e.g. ✓ Bad e.g. ☹ for north pole & south pole.

<u>Fact</u> (Milnor) For any Lie group G, there exists a contractible space EG s.t. $G \curvearrowright EG$ is free.

e.g. $G = S^1$, then $EG = \underset{\text{contractible!}}{S^\infty} = \left\{ (z_1, z_2, \dots) \in \mathbb{C}^\infty \;\middle|\; \begin{array}{l} \text{all but finite} = 0 \\ \sum |z_i|^2 = 1 \end{array} \right\}$

$\Longrightarrow$ Starting from $G \curvearrowright M$ (not nec. free), one can cook up a new mfd X s.t. $G \curvearrowright X$ free and <u>more importantly, $M \simeq X$.</u>

$$X = M \times EG \quad \text{and} \quad g \cdot (m, x) = (g \cdot m, g^{-1} x)$$

diagonal action

This is called "Borel construction".

Thm For $G \curvearrowright M$ where G is a cpt* Lie group and action is free, then $M/G :=$ quotient manifold with equ relation

$$x \sim y \quad \text{iff} \quad y = g\cdot x \text{ for some } g \in G$$ ← modulo the orbit space

is a mfd of $\dim M/G = \dim M - \dim G$.

e.g. $\mathbb{R}^{n+1}\setminus\{0\}/\mathbb{R}^* \; (\simeq \mathbb{RP}^n)$, $\;S^{2n+1}/S^1 \; (\simeq \mathbb{CP}^n)$, $\;\mathbb{C}^n\setminus\{0\}/\mathbb{T}^n \; (\simeq \mathbb{R}^n_{>0})$

- $G \curvearrowright M$ is called <u>transitive</u> if there is only one orbit space.

Thm If $G \curvearrowright M$ is transitive, where G is a cpt Lie group, then

$$M = G\cdot\{x\} \overset{\text{algebra}}{\simeq} G/G_x .$$

any $x \in M$

Such M is called a <u>homogeneous space</u> (齐性空间)

(why notation G/G_x makes sense?)

e.g. (From Exe, $O(n) = \{ A \in GL(n) \mid AA^T = A^TA = \mathbb{1} \}$ orthogonal matrix

is a cpt Lie group of dim $= \frac{n(n-1)}{2}$.)

Consider

Grassmannian (over $\mathbb{R}$) $\to$ $Gr_{\mathbb{R}}(k,n) = \{$ k-dim linear subspaces of $\mathbb{R}^n \}$

- $O(n)$ acts on $Gr_{\mathbb{R}}(k,n)$ transitively.
- For $\mathbb{R}^k \times \{0\} \in Gr_{\mathbb{R}}(k,n)$, $O(n)_{\mathbb{R}^k\times\{0\}} = \{ A \in O(n) \mid A\cdot(\mathbb{R}^k\times\{0\}) = \mathbb{R}^k \times \{0\} \}$

$$\underbrace{\begin{array}{c} {\scriptstyle k\times k} \\ \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \\ {\scriptstyle (n-k)(n-k)} \end{array}}_{A} \begin{pmatrix} \overset{\begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}}{e} \\ 0 \end{pmatrix} = \begin{pmatrix} Xe \\ Ze \end{pmatrix} \in \mathbb{R}^k \times \{0\} \Rightarrow Z = 0$$

$$AA^T = \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} \begin{pmatrix} X^T & 0 \\ Y^T & W^T \end{pmatrix} = \begin{pmatrix} XX^T & YW^T \\ WY^T & WW^T \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{k\times k} & 0 \\ 0 & \mathbb{1}_{(n-k)\times(n-k)} \end{pmatrix}$$

$\Rightarrow Y = 0$ and $X \in O(k)$ and $W \in O(n-k)$

Therefore, $O(n)_{\mathbb{R}^k \times \{0\}} = O(k) \times O(n-k)$.

<u>Thm</u> + Rmk above imply $Gr_{\mathbb{R}}(k,n) \simeq O(n) / O(k) \times O(n-k)$

which is a manifold of dim $= \frac{n(n-1)}{2} - \left(\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \right)$

$$= \frac{1}{2}\left(-2k^2 + 2nk\right) = k(n-k).$$

In particular, when $k=1$, $Gr_{\mathbb{R}}(1,n) = \mathbb{R}P^{n-1} = \left(\left\{ \begin{array}{l} \text{lines in } \mathbb{R}^n \\ \text{through } 0 \end{array} \right\} \right)$

<u>Rmk</u> Over $\mathbb{C}$, argument above works and

$$Gr_{\mathbb{C}}(k,n) \simeq \underset{\leftarrow \text{ real dim} = n^2}{U(n)} / U(k) \times U(n-k)$$ mfd of $\dim_{\mathbb{C}} = k(n-k)$.

In particular, $Gr_{\mathbb{C}}(1,n) = \mathbb{C}P^{n-1} \left(= \left\{ \begin{array}{l} \text{cpx lines in } \mathbb{C}^n \\ \text{through } 0 \end{array} \right\} \right)$.